Exercise 4

[1] Concisely describe the differences between Fourier transform and Fourier series expansion.

[2] (Optional: if you think you’ve done before and confident, you can skip): Demonstrate the orthogonality

\[
\int_0^a \sin \left( \frac{2\pi n}{a} x \right) \sin \left( \frac{2\pi m}{a} x \right) dx = 0 \quad \text{unless} \quad n = m
\]

\[
\int_0^a \cos \left( \frac{2\pi n}{a} x \right) \cos \left( \frac{2\pi m}{a} x \right) dx = 0 \quad \text{unless} \quad n = m
\]

\[
\int_0^a \sin \left( \frac{2\pi n}{a} x \right) \cos \left( \frac{2\pi m}{a} x \right) dx = 0 \quad \text{for all} \quad n, m
\]

[3] Consider a periodic domain \( 0 \leq x \leq 4\pi \). Periodic means the left boundary and the right boundary are connected to each other.

(a) Solve for \( E \) from Gauss’s law

\[
\frac{\partial E(x)}{\partial x} = \rho(x)
\]

when \( \rho(x) = A \cos kx \). Here, we set \( E(0) = 0 \) as a boundary condition. What condition should \( k \) satisfy?

(b) Solve Eq.\( (1) \) using Fourier transform for \( \rho(x) \) and \( E(x) \), and then inverse Fourier transform. Then apply \( \rho(x) = A \cos kx \).

[4] (Schwartz 2-3) Find the energy stored in a uniform, spherical charge distribution of radius \( R \) and total charge \( Q \). For now, you can estimate the stored energy by

\[
U = \frac{1}{8\pi} \int_{all-space} E^2 dV.
\]
\[ f(x) = \sum A_n \sin(2\pi nx/a) \]

**Figure 1:** An example of Fourier series expansion within a domain \(0 \leq x \leq a\).

\[ f(x) = \exp(-x^2) \]

**Figure 2:** An example of Fourier transform when \(k\) is no longer integer and the domain \(a \rightarrow \infty\).