Exercise 2

[1] (Gauss’ theorem) Obtain the surface integral
\[ \int_S \mathbf{A} \cdot \mathbf{n} dS \]
for a vector \( \mathbf{A} = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k} \) on the surface \( S \). Here, the sphere \( S \) is given at the radius \( a \) with the origin at the center. Hint: use spherical polar coordinate system \( x = r \sin \theta \cos \varphi \), \( y = r \sin \theta \sin \varphi \), and \( z = r \cos \theta \) when you do the volume integral.

[2] (Stokes’ theorem) Estimate the line integral
\[ \oint \mathbf{A} \cdot d\mathbf{l} \]
for \( \mathbf{A} = (x^2 + y^2) \mathbf{i} + (x^2 + 2z) \mathbf{j} + 2y \mathbf{k} \) along a circle \( x^2 + y^2 = 4 \) on the \( x - y \) plane.

[3] (Wave) From Maxwell’s equation in a vacuum, derive wave equations
\[ \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]
and
\[ \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}. \]
Are these transverse or longitudinal waves?

**optional** Can transverse waves always be electromagnetic? Can the electromagnetic waves always be transverse? Why?

[4] (Wave) Let us take a one dimensional limit of the electromagnetic wave
\[ \frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \]
Demonstrate that the \( E(x, t) = f(x - ct) \) can be a general solution. Does \( E(x, t) = f(x - ct) \) move to the right or left, and why? How about \( E(x, t) = f(x + ct) \)? What is the phase velocity of the wave?