Exercise 1

[1] Write out the Maxwell’s equations in a differential form. Use the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ on the left side of the Maxwell’s equations. The answers can be given either in SI units or Gaussian units, but need to be consistent.

Change the differential form of the Maxwell’s equations into the integral form using the Gauss’s divergence theorem

$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \int_V (\nabla \cdot \mathbf{F}) dV$$

which relates a surface integral and a volume integral, and the Stoke’s theorem

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

which relates a line integral and a surface integral. Here, $\mathbf{F}$ is an arbitrary vector, $d\mathbf{l}$, $dS$, and $dV$, are line, surface, and volume elements, respectively. The unit vector normal to the surface element $dS$ is given by $\mathbf{n}$.

Explain the physical meaning of each of the Maxwell’s equations. Three out of the four relations have names. What are they called?

By introducing the vector potential $\mathbf{A}$ and the scalar potential $\Phi$, demonstrate that the differential form of Maxwell’s equations can be reduced to a set of four equations, which is composed of one scalar equation and one vector equation. By introducing one of the gauge conditions (the Lorentz gauge condition), show that all the four equations can be expressed by the Poisson-type equations.

[2] Estimate $\nabla \cdot \mathbf{r}$ for $\mathbf{r} = (x, y, z)$.

[3] Demonstrate the following vector identities.\(^1\)

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \]

\[ \nabla \times \nabla f = 0 \]

\[ \nabla \cdot (\nabla \times \mathbf{A}) = 0 \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

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\(^1\)The first one referred to as a “back-cab” rule (imagine a taxi going backward).