• $A \cdot x = b$ left over.

• From numerical integration to solving ODE.

• ODE solve by Euler method.

• Single charged particle motion in electromagnetic field.

• Check $E \times B$ drift in 2D geometry.
Let us look into examples of ODEs

- Celestial mechanics. Planet and satellite motion.
- Single charged particle motion in electromagnetic field.
- Guiding center motion. Magnetic field lines. Pendulum equation.
- An example of ordinary differential equation that can be solved analytically, e.g. a harmonic oscillator

\[ m \frac{d^2 x}{dt^2} = -kx \]

solution is given by \( x(t) = A \sin \left( \sqrt{\frac{m}{kt}} + \delta_0 \right) \). The amplitude \( A \) and \( \delta_0 \) need to be determined by IC.
By the way how do we solve $m\ddot{x} = -kx$?

- We reduce the 2nd order ODE to a system of 1st order ODE $\times 2$ (simultaneous ODE).

\[
m\frac{dv}{dt} = -kx,
\]

\[
\frac{dx}{dt} = v.
\]

In other words, integrate

\[
v(t) = v_0 + \left(-\frac{k}{m}\right) \int_0^t x(t')dt'
\]

and

\[
x(t) = x_0 + \int_0^t v(t')dt'
\]

at the same time.
Time advancing dynamical equations is nothing but numerical integration

- For a simplest example, consider a charged particle in 1D system with a constant electric field

\[ m \frac{dv}{dt} = qE(x) \]

\[ \frac{dx}{dt} = v \]

- By a discretization, numerical integration by the most primitive Euler method reads

\[ \frac{v^{(k+1)} - v^{(k)}}{\Delta t} = \frac{q}{m} E \left[ x^{(k)} \right] \rightarrow v^{(k+1)} = v^{(k)} + \Delta t \frac{q}{m} E \left[ x^{(k)} \right] \]

\[ \frac{x^{(k+1)} - x^{(k)}}{\Delta t} = v^{(k)} \rightarrow x^{(k+1)} = x^{(k)} + \Delta tv^{(k)} \]
Simplest algorithm to solve ODE is given

- To start with (See Chapter 10.1 of Cheney and Kincaid.)

\[ \frac{dx}{dt} = f(t, x) \]

where \( x = x(t) \) and \( x(a) = x_a \) is given.

- Euler method is based on Taylor expansion

\[ x(t + h) = x(t) + hx'(t) + O(h^2) \]

and then

\[ x(t + h) = x(t) + hf(t, x) \]

- We will do 2nd order later. First let us get a feeling about integrating the equation of motion.

- First example. One 1st order ODE. Let us solve

\[ \frac{dx}{dt} = -x \]

with \( x(0) = 1.0 \) as an IC. We know the solution to be \( x(t) = \exp(-t) \).
• An example with $h = 0.1$ (very coarse to emphasize the error). Red curve is the analytical solution $x(t) = \exp(-t)$.

\[ \text{To plot log scale in xmgrace, type xmgrace -log y fort.1 fort.2 for example.} \]
Now note that if we analytically solve
\[ x(t + h) = x(t) + hf(x) = x(t) + h(-x) \]
or
\[ x_{k+1} = x_k + h(-x_k), \]
we obtain
\[ x_{k+1} = (1 - h)x_k = (1 - h)^2x_{k-1} = ... = (1 - h)^k. \]

If we set \( kh = t = \text{const} \) and take the limit of \( h \to 0 \) (and thus \( k \to \infty \))
\[ x(t) = \lim_{h \to 0} (1 - h)^k = \lim_{h \to 0} (1 - h)^{t/h} = \lim_{h \to 0} [(1 - h)^{1/h}]^t = (e^{-1})^t = e^{-t}. \]

Since we are not able to take \( h \to 0 \) computationally, there is always an error.
A pseudo-code for the Euler method is given

- An example below is a pseudo-code which can be applied for any language (f90 or C, etc.).

\[
h \leftarrow t/n \\
t \leftarrow 0.0 \\
x \leftarrow 1.0 \\
\textbf{for } k = 1 \text{ to } n \textbf{ do} \\
\quad x = x + hf(t, x) \\
\quad t = t + h \\
\quad \text{printf } k, t, x \\
\textbf{end for}
\]
Single particle motion with Lorentz force

- Single charged particle motion in an magnetic field $\mathbf{B}$ is given by

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B})$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

where $m$ is the mass, $q$ is the charged. We set $E = 0$ for now.

- After normalization by $\bar{B} = B/B_0$, $\bar{t} = (qB_0/m)t = \Omega_c t$, $\bar{x} = x/a$, and $\bar{v} = v/a\Omega_c$ (the bars denote normalized values), \(^a\)

$$\frac{d\bar{\mathbf{v}}}{dt} = \bar{\mathbf{v}} \times \bar{\mathbf{B}}$$

$$\frac{d\bar{\mathbf{x}}}{dt} = \bar{\mathbf{v}}.$$

- In 3D Cartesian it reads

$$\frac{d}{dt} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} v_y B_z - v_z B_y \\ v_z B_x - v_x B_z \\ v_x B_y - v_y B_x \end{pmatrix} \quad \text{and} \quad \frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

\(^a\)All the values are now order of unity.
• Taking $z$ symmetry, and $B_x = 0, B_y = 0$ it further simplifies to

$$\frac{dv_x}{dt} = v_y B_z$$

$$\frac{dv_y}{dt} = -v_x B_z$$

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

• To see a simple cyclotron motion $B_z(x, y)$ can take the form

$$B_z(x, y) = 1.0$$

• You can also try with magnetic field inhomogeneity (see $\nabla B$ drift).

• Does kinetic energy $E_k \bar{v}_x^2/2 + \bar{v}_y^2/2$ conserve?
Let us examine $E \times B$ drift

- In the presence of electric field

\[ m \frac{dv}{dt} = q(E + v \times B) \]

\[ \frac{dx}{dt} = v \]

where normalization is given by $\bar{E} = E/(B_0 a \Omega_c)$. On 2D plane (we can set $E = E_x \hat{x}$)

\[ \frac{dv_x}{dt} = v_y B_z + E_x \]

\[ \frac{dv_y}{dt} = -v_x B_z \]

\[ \frac{dx}{dt} = v_x \]

\[ \frac{dy}{dt} = v_y \]

- Is $E \times B$ drift charge dependent? Try changing the charge sign $q \to -q$.

- What if $E$ is time dependent? (polarization drift).
Now let us recall how we derived $E \times B$

- Imagine $E$ in $\hat{x}$ direction. When the charge is positive the ion is accelerated on the right side. This gives a larger gyroradius at the right that the left. The motion will be spiral.

- If we average over many gyro-periods, the average acceleration is zero.

\[
m\dot{v} = 0 = q(E + v \times B)
\]

Taking the cross product with $B$

\[
E \times B = B \times (v \times B) = vB^2 - B(v \cdot B)
\]

and assuming $v_d \cdot B = 0$, we obtain the $E \times B$ drift velocity,

\[
v_d = \frac{E \times B}{B^2}
\]
Particle motion in magnetic mirror is plotted

- Employing the cylindrical coordinate the magnetic field here is given by
  \[ B_r = -rz \quad B_z = 1 + z^2 \]
  which satisfies \( \nabla \cdot \mathbf{B} = 0 \). Depending on the pitch angle, the particle trajectories experience trapped (or untrapped=passing) behaviors.
Charged particles feels mirror force when it moves into strong magnetic field

- In the previous cylindrical configuration, consider a force in z direction

\[ F_z = q|v \times B|_z = qv_\perp B_r \]

Since

\[ \nabla \cdot B = \frac{1}{r} \frac{\partial}{\partial r} (rB_r) + \frac{\partial B_z}{\partial z} = 0, \]

we obtain \( B_r = -(r/2)\partial_z B_z \).

- Considering a particle whose center of gyration is on the z-axis,

\[ F_z = -qv_\perp \frac{r}{2} \frac{\partial B_z}{\partial z} = -qv_\perp \frac{\rho_L}{2} \frac{\partial B_z}{\partial z} = -\frac{mv_\perp^2}{2B} \frac{\partial B_z}{\partial z} = -\mu \frac{\partial B_z}{\partial z} \]

where the Larmor radius is given by \( \rho_L = mv_\perp / qB \)
and the magnetic moment is given by \( \mu = mv_\perp^2 / 2B \).
$V \times B$ is solved in a dipole magnetic field

- Lorentz force term is pushed in a Cartesian $(x, y, z)$ coordinate.

$$m \frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}$$

$$\frac{dx}{dt} = \mathbf{v}.$$

$$B_r = -\frac{\mu_0 M \sin \lambda}{2\pi \frac{r^3}{r^3}} \quad B_\lambda = -\frac{\mu_0 M \cos \lambda}{4\pi \frac{r^3}{r^3}} \quad B_\phi = 0$$
Higher order integral method

- Modified Euler method (2nd order method)

\[
x(t + h/2) = x(t) + \frac{h}{2} f(t, x)
\]

\[
x(t + h) = x(t) + hf(t + h/2, x(t + h/2))
\]

- Fourth order Runge-Kutta method

\[
x(t + h) = x(t) + \frac{1}{6} (K_1 + K_2 + K_3 + K_4)
\]

\[
K_1 = hf(t, x)
\]

\[
K_2 = hf(t + h/2, x + K_1/2)
\]

\[
K_3 = hf(t + h/2, x + K_2/2)
\]

\[
K_4 = hf(t + h, x + K_3)
\]
Summary of today’s discussions

• Gaussian-elimination for $A \cdot x = b$.

• ODE solve by the Euler method.

• Single charged particle motion in electromagnetic field.

• 3/5 and 3/12 will continue on ODE. Guiding centers, chaos. Satellite trajectory.

• Modified Euler (2nd order) method.