• Multi-dimensional Poisson equation.

• Finite difference and Finite element method.
We encounter multi dimensional diffusion problems in practical application

- Two dimensional Laplacian operator for Poisson
  \[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y) \]

- Another example on two dimensional diffusion
  \[ \frac{\partial u}{\partial t} = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \]
2D Elliptic equation can be solved by finite difference method

- Let us consider a 2D elliptic equation in general

\[ \nabla^2 u(x, y) + f(x, y)u(x, y) = g(x, y). \]

Assuming a uniform grid in x and y, the Laplacian operator will be given by

\[ \nabla^2 u = \frac{1}{h^2} \left[ u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y) \right] \]

If we write \( x_i = i \cdot h \) and \( y_j = j \cdot h \) \((i, j \geq 0)\), with the index

\[ (\nabla^2 u)_{i,j} = \frac{1}{h^2} (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) \]

and therefore the 2D elliptic equation looks like

\[ -u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + (4 - h^2 f_{i,j}) u_{i,j} = h^2 g_{i,j} \]
Consider a domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and $h = 1/4$, and taking Dirichlet boundary conditions

\[-u_{21} - u_{01} - u_{12} - u_{10} + (4 - h^2 f_{11}) u_{11} = h^2 g_{11} \]
\[-u_{31} - u_{11} - u_{22} - u_{20} + (4 - h^2 f_{12}) u_{21} = h^2 g_{21} \]
\[-u_{41} - u_{21} - u_{32} - u_{30} + (4 - h^2 f_{13}) u_{31} = h^2 g_{31} \]
\[-u_{22} - u_{02} - u_{13} - u_{11} + (4 - h^2 f_{12}) u_{12} = h^2 g_{12} \]
\[-u_{32} - u_{12} - u_{23} - u_{21} + (4 - h^2 f_{22}) u_{22} = h^2 g_{22} \]
\[-u_{42} - u_{22} - u_{33} - u_{31} + (4 - h^2 f_{32}) u_{32} = h^2 g_{32} \]
\[-u_{23} - u_{03} - u_{13} - u_{12} + (4 - h^2 f_{13}) u_{13} = h^2 g_{13} \]
\[-u_{23} - u_{13} - u_{23} - u_{22} + (4 - h^2 f_{23}) u_{23} = h^2 g_{23} \]
\[-u_{23} - u_{23} - u_{33} - u_{32} + (4 - h^2 f_{33}) u_{33} = h^2 g_{33} \]

Five stencil relations are satisfied for each point.
• Can be case into matrix equation

\[
\begin{pmatrix}
4 - h^2 f_{11} & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 - h^2 f_{21} & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 - h^2 f_{31} & 0 & 0 & -1 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 - h^2 f_{33}
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7 \\
U_8 \\
U_9
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6 \\
b_7 \\
b_8 \\
b_9
\end{pmatrix}
\]
Solutions to the example problem (15.3 Cheney and Kincaid),

\[ \nabla^2 u - \frac{1}{25} u = 0 \]

\[ u = \cosh(x/5) + \cosh(y/5) \] on the boundary.
A few more notes on Laplacian differential operator is given

- There is also a nine-point scheme.

\[
(\nabla^2 u)_{i,j} = \frac{1}{6h^2} (4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} - u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 20u_{i,j})
\]

- The cylindrical coordinate Laplacian operator

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}
\]

is given by

\[
\frac{1}{r_i} \frac{1}{\Delta r} \left( r_{i+1/2} \frac{u_{i+1,j} - u_{i,j}}{\Delta r} - r_{i-1/2} \frac{u_{i,j} - u_{i-1,j}}{\Delta r} \right) + \frac{1}{r_i^2} \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta \theta^2}.
\]