• Introduce partial differential equation.

• Numerical solution of diffusion equation with spatially varying D.

• Modeling of turbulence transport.

• Advection-diffusion equation (Burger’s equation).

• Cole-Hopf transformation.
Partial differential equations can be solved numerically

- Let us start with 1D diffusion equation.
  \[
  \frac{\partial f}{\partial t} = D \frac{\partial^2 f(x, t)}{\partial x^2}.
  \] (1)

- Given an appropriate initial condition, the solution is given by
  \[
  f(x) = \frac{1}{\sqrt{4\pi Dt}} \exp \left[ \frac{-x^2}{4Dt} \right].
  \] (2)

- Now discretize Eq.(1):
  \[
  \frac{f_{j}^{i+1} - f_{j}^{i}}{\Delta t} = D \frac{f_{j+1}^{i} - 2f_{j}^{i} + f_{j-1}^{i}}{(\Delta x)^2}.
  \] (3)

We can solve Eq.(3) numerically. Examine the numerical solution by comparing it to the analytical one Eq.(2).
Monte-Carlo simulation and diffusion simulation compared

- Explicit scheme reads

\[ f_j^{i+1} = D \frac{\Delta t}{(\Delta x)^2} (f_{j+1}^i + f_{j-1}^i) + \left( 1 - 2D \frac{\Delta t}{(\Delta x)^2} \right) f_j^i \]  

(a) \hspace{1cm} (b) \hspace{1cm} (c)
**Diffusion equations with spatially varying $D$ is solved**

- The random step of Monte-Carlo simulation is given by $(-0.5 < r < 0.5)$
  
  $$ r = r \cdot (ax + b) $$  
  
  (5)

- Correspondingly the diffusion is given by

  $$ D = D_{const} \cdot (ax + b)^2 $$

  (6)
A basic picture of coupling transport in 2D-divertor code and turbulence sim. is given

\[ \nabla n, \nabla T \leftrightarrow D = -\Gamma \nabla n \quad \text{[Transport] (B2)} \]

\[ \nabla n, \nabla T \leftrightarrow \chi = -Q \nabla T \quad \text{[Turbulence] (DW)} \]

Needs \( D \) & \( \chi \)

- The basic idea behind is to replace the fast time scale oscillatory and repetitive turbulence vortex dynamics by the "\( \nabla^2 \)" operator.
Turbulence simulation: 4-field
Hasegawa-Wakatani model is employed

\[
\frac{d}{dt} \nabla^2 \varphi = \nabla J_\parallel - \mathcal{K}(p_e + p_i)
\]

\[
\frac{dn}{dt} = -\omega_n \frac{\partial \varphi}{\partial y} + \nabla J_\parallel - \mathcal{K}(p_e - \varphi)
\]

\[
\frac{3}{2} \frac{dT_e}{dt} = -\frac{3}{2} \omega_t \frac{\partial \varphi}{\partial y} + \nabla (J_\parallel - q_\parallel) - \mathcal{K}\left(\frac{5}{2} T_e + p_e - \varphi\right)
\]

\[
\frac{3}{2} \frac{dT_i}{dt} = -\frac{3}{2} \omega_i \frac{\partial \varphi}{\partial y} + \nabla J_\parallel + \mathcal{K}\left(\frac{5}{2} T_i - p_e + \varphi\right)
\]

- Here, \( p_e = n + T_e, \mathcal{K} = \omega_B \partial_y \), and \( d_t = \partial_t + b \times \nabla \varphi \cdot \nabla \).

- Electrostatic Ohm’s law, \( \eta J_\parallel = \nabla (n + 1.71T_e - \varphi) \). The \( \nabla J_\parallel \) operator incorporates the dissipation coupling parameter \( D_{HW} = \left(C_s/\nu_{ei}L_\perp\right)(m_i/m_e)k_\parallel^2L_\perp^2 \). Phase difference “\( n - \varphi \)” explicit in HW: large in collisional regimes.

- Heat flux, \( -q_\parallel = 1.6 \nabla T_e + 0.71J_\parallel \).
An example of resistive drift-wave turbulence simulation is shown

- Equilibrium quantities larger on left, $\nabla n, \nabla T$ goes like $\downarrow$ giving net radial fluxes: left $\rightarrow$ right.
- ZF regulates. Note the finite phase difference between $n$ and $\varphi$. 
Turbulence flux is time averaged for $D, \chi$’s

Time averaged flux: $\langle \Gamma \rangle_t = (1/t) \int_0^t \Gamma(t') dt'$.
Note uncertainties in $\langle \Gamma \rangle_t$. 
Drift wave dynamics:
Transport increase with $\nabla n$ and $\nabla T$

- (Right) Transport coefficients as functions of $\nabla n$, $\nabla T_i$, and $\nabla T_e$, obtained from turbulence simulation. a

- (Left) A conceptual diagram: role of turbulence is to smooth out inhomogeneity (black $\rightarrow$ red).

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aNote that the three values are all independently parameterized, $\omega_n$, $\omega_t$, $\omega_i$ also being independent.
System of diffusion equations is solved

- Consider simultaneous evolution of plasma density and temperature.

\[
\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial x^2}
\]

\[
\frac{\partial T_i}{\partial t} = \chi_i \frac{\partial^2 T_i}{\partial x^2} + D_{Ti} \frac{\partial^2 n}{\partial x^2}
\]

\[
\frac{\partial T_e}{\partial t} = \chi_e \frac{\partial^2 T_e}{\partial x^2} + D_{Te} \frac{\partial^2 n}{\partial x^2}
\]
An example of $n$, $T_i$, and $T_e$ profiles at the steady state is shown

- Space and time dependent transport coefficients are employed.$^a$
- Our focus is on two dotted curves, constant $D, \chi$’s vs “space and time dependent” $D, \chi$’s.$^b$

$^a$Neumann (constant flux) boundary conditions are taken at the core (left) side.
$^b$Solid lines include the effect of dissipation coupling parameter, $D_{HW} = (C_s/\nu e_i L_\perp)(m_i/m_e)k^2_\parallel L^2_\perp$, which is outside the scope for the moment.
Reduced transport model employed in 2D transport calculation incorporates self-consistent $E_r$ evolution

\[
\partial_t n + \nabla \cdot (-D \nabla n) = S^n
\]

\[
\partial_t (m_i n V_i) = -\nabla p_i - \nabla \cdot \Pi_i + Zen_i (E + V_i \times B) + R + S^m
\]

\[
\partial_t \left( \frac{3}{2} n T_e \right) = -\nabla \cdot \left( \frac{5}{2} n T_e V - \chi_e \nabla T_e \right) - enE \cdot V_e + R \cdot V_e + k (T_i - T_e) + S^H_e
\]

\[
\partial_t \left( \frac{3}{2} n T_i \right) = -\nabla \cdot \left( \frac{5}{2} n T_i V_i + \frac{m_i n}{2} V_i^2 V_i + \Pi_i \cdot V_i - \chi_i \nabla T_i \right) + ZenE \cdot V_i - R \cdot V_i + k (T_e - T_i) + S^H_i
\]

\[
\nabla \cdot J = 0, \quad V_{\perp,i,e} = \frac{B \times \nabla \phi}{B^2} \pm \frac{B \times \nabla p_{i,e}}{enB^2}
\]

\(^{a}\)F.L.Hinton and Y.B.Kim, Nuclear Fusion 34, 899 (1994).
Advection (convection) equation is given by factorization of the wave equation

- We focus on the advection equation
  \[
  \frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x}.
  \]
  where \( u = u(x, t) \) and \( c = c(x, t) \).\(^a\)

- The model equation can be approximated by forward time and central difference
  \[
  \frac{1}{k} [u(x, t + k) - u(x, t)] = -c \frac{1}{2h} [u(x + h, t) - u(x - h, t)]
  \]
  which gives
  \[
  u(x, t + k) = u(x, t) - \sigma \frac{1}{2} [u(x + h, t) - u(x - h, t)]
  \]
  where \( \sigma = (k/h)c(x, t) \).

\(^a\)The characteristics.
A nonlinear wave equation, advection-diffusion (Burger’s equation) equation is introduced

- The advection-diffusion equation (Burger’s equation) can be regarded as one dimensional limit of Navier-Stokes equation:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}
\]

The Burger’s equation possess analytical solution (from Cole-Hopf transformation) and thus can be a useful test-bed to check the numerical schemes for the further application in Navier-Stokes system with higher dimension.

- Let us give an initial condition at \( t = 0 \),

\[ u(x, 0) = \sin (\pi x). \]

- Fixed point boundary conditions are given by

\[ u(0, t) = u(1, t) = 0. \]

- We apply both the forward time central difference scheme and the upwind.
Burgers’ equation can be solved numerically

By the forward time central difference scheme (explicit scheme), the model equation is approximated by

\[
\frac{1}{k} [u(x, t + k) - u(x, t)] + u(x, t) \frac{1}{2h} [u(x + h, t) - u(x - h, t)] = \nu h^2 [u(x + h, t) - 2u(x, t) + u(x - h, t)]
\]

which reads

\[
u h^2 [u(x + h, t) - 2u(x, t) + u(x - h, t)]
\]

\[
\frac{\nu k}{h^2} [u(x + h, t) - 2u(x, t) + u(x - h, t)] + u(x, t) - \frac{k}{2h} u(x, t) [u(x + h, t) - u(x - h, t)]
\]
We see steepening in Burgers’

- As a reminder consists of two terms, convective and diffusive term.
  \[
  \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}
  \]
  - The larger the slope \((\partial_x u)\), faster it moves.
  - Three cases \(\nu = 1.0 \times 10^{-3}\), \(\nu = 2.0 \times 10^{-2}\), and \(\nu = 1.0 \times 10^{-1}\) (extremely diffusive) are shown.

![Graphs](image-url)
Cole-Hopf transformation is given

- Start from Burgers equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}
\]

- Letting \( u = \partial_x \Phi \), we obtain

\[
\frac{\partial^2 \Phi}{\partial x \partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right)^2 = \nu \frac{\partial^3 \Phi}{\partial x^3}
\]

thus

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 = \nu \frac{\partial^2 \Phi}{\partial x^2}
\]

- We want to eliminate the 2nd term by letting \( \Phi = -2\nu ln\theta \), thus each term becomes

\[
\frac{\partial \Phi}{\partial t} = -2\nu \frac{1}{\theta} \frac{\partial \theta}{\partial t},
\]

\[
\frac{\partial \Phi}{\partial x} = -2\nu \frac{1}{\theta} \frac{\partial \theta}{\partial x} \rightarrow \left( \frac{\partial \Phi}{\partial x} \right)^2 = 4\nu^2 \frac{1}{\theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2,
\]
\[
\frac{\partial^2 \phi}{\partial x^2} = 2\nu \frac{1}{\theta^2} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} - 2\nu \frac{1}{\theta} \frac{\partial^2 \theta}{\partial x^2},
\]

- Substituting all the terms we obtain

\[-2\nu \frac{1}{\theta} \frac{\partial \theta}{\partial t} + \frac{1}{2} 4\nu^2 \frac{1}{\theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 = 2\nu^2 \frac{1}{\theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 - 2\nu^2 \frac{1}{\theta} \frac{\partial^2 \theta}{\partial x^2},\]

or simply

\[
\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2},
\]

which is nothing but a diffusion equation whose solution is known.
Summary of discussions

• Numerical solution of diffusion equation with spatially varying D.
• Modeling of turbulence transport.
• Advection-diffusion equation (Burger’s equation).
• Cole-Hopf transformation.

• Analytical solution of diffusion; (1) Fourier transform (2) Similarity method.
• Courant, Fredrichs, and Lewi (CFL) analysis of numerical stability.
Parametrization:
Least square fitting is employed for $D$, $\chi$’s

- The general form of the model is given by

$$y = \sum_{k=1}^{M} a_k X_k(x).$$

Here, $x$ is for $\nabla n$ etc, $y$ for $D$ and $\chi$’s, and $X_k$’s for Chebyshev polynomial bases.

- Our task is to pick the best parameters to minimize

$$\chi^2 = \sum_{i=1}^{N} \left[ \frac{y_i - \sum_{k=1}^{M} a_k X_k(x_i)}{\sigma_i} \right]^2$$

- Singular value decomposition employed.
• An extremely long sampling time (up to $10^5 L_\perp / C_s$) is taken. Intermittency is inherent due to dissipation.

• Standard deviation $\sigma_i$ can be obtained from the width.